Determinant Formulas for Matrix Model Free Energy

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Abstract

The paper contains a new non-perturbative representation for subleading contribution to the free energy of multicut solution for hermitian matrix model. This representation is a generalisation of the formula, proposed by Klemm, Marino and Theisen for two cut solution, which was obtained by comparing the cubic matrix model with the topological B-model on the local Calabi-Yau geometry $\widehat{II}$ and was checked perturbatively. In this paper we give a direct proof of their formula and generalise it to the general multicut solution.

1   Introduction

An interest to the multicut solutions to matrix models was inspired by the studies in $\mathcal{N}=1$ supersymmetric gauge theories due to Cachazo, Intrilligator and Vafa [1] and Dijkgraaf, Vafa [2, 3, 4] who proposed to calculate the nonperturbative superpotentials of $\mathcal{N}=1$ SUSY gauge theories in four dimensions using matrix models technique. This $\mathcal{N}=1$ theories contains the multiplet of $\mathcal{N}=2$ SUSY gauge theories but with nontrivial tree superpotential. The nonperturbative superpotential could be obtained from the partition functions of the one-matrix model (1MM) in the leading order in $1/N$, N being the matrix size. Higher genus corrections are identified with certain holomorphic couplings of gauge theory to gravity.

The authors of [5] proposed a new anzatz for $\mathcal{F}_1$ in the two-cut case (with absent double points) and made a perturbative check. Their formula in fact comes from the correspondence between the so called topological B-model on the local Calabi-Yau geometry $\widehat{II}$ and the cubic matrix model. Here we give complete proof of this formula and generalize it to the multi-cut case.

We start with definition of the matrix integral and introduce all relevant constructions. For a complete review of the subject, see [6] and references there in. Consider the hermitian 1-matrix model:

$$\int_{N\times N} DX\, e^{-\frac{1}{\hbar}trV(X)} = e^{\mathcal{F}}, \tag{1}$$

where $V(X) = \sum_{n\geq 1} t_n X^n$, $\hbar = \frac{t_0}{N}$ is a formal expansion parameter, the integration goes over the $N\times N$ matrices, $DX \propto \prod_{ij} dX_{ij}$

The topological expansion of the Feynman diagrams series is then equivalent to the expansion in even powers of $\hbar$ for

$$\mathcal{F} \equiv \mathcal{F}(\hbar, t_0, t_1, t_2, \dots) = \sum_{h=0}^{\infty} \hbar^{2h-2}\mathcal{F}_h, \tag{2}$$

Customarily $t_0 = \hbar N$ is the scaled number of eigenvalues. We assume the potential $V(p)$ to be a polynomial of the fixed degree $m+1$.

The averages, corresponding to the partition function (1) are defined as usual:

$$\langle f(X)\rangle = \frac{1}{Z}\int_{N\times N} DX\, f(X)\exp\left(-\frac{1}{\hbar}trV(X)\right) \tag{3}$$

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and it is convenient to use their generating functionals: the one-point resolvent

$$W(\lambda) = \hbar \sum_{k=0}^{\infty} \frac{\langle trX^k \rangle}{\lambda^{k+1}}. \tag{4}$$

as well as the s-point resolvents ($s \geq 2$)

$$W(\lambda_1, \ldots, \lambda_s) =$$

$$\hbar^{2-s} \sum_{k_1, \ldots, k_s = 1}^{\infty} \frac{\langle trX^{k_1} \cdots trX^{k_s} \rangle_{\text{conn}}}{\lambda_1^{k_1+1} \cdots \lambda_s^{k_s+1}} =$$

$$\hbar^{2-s} \left\langle tr\frac{1}{\lambda_1 - X} \cdots tr\frac{1}{\lambda_s - X} \right\rangle_{\text{conn}} \tag{5}$$

The genus expansion of the resolvent has the form

$$W(\lambda_1, \ldots, \lambda_s) = \sum_{h=0}^{\infty} \hbar^{2h} W_h(\lambda_1, \ldots, \lambda_s), \quad s \geq 1, \tag{6}$$

It satisfies the loop equation [7, 8]:

$$[V'(x)W(x)]_- = W(x)^2 + \hbar^2 W(x, x), \tag{7}$$

where $[...]_-$ is the projector on the negative powers. In genus zero, loop equations have the solution

$$W_0(\lambda) = \frac{1}{2}(V'(\lambda) - y) \tag{8}$$

$$y^2 = V'(\lambda)^2 + 4P_{m-1}(\lambda), \tag{9}$$

where P_{m-1} is an arbitrary polynomial of degree $m - 1$. If the curve (9) has n cuts, it can be represented in terms of branching points μ_α

$$y \equiv M(\lambda)\tilde{y} \equiv M(\lambda)\sqrt{\prod_{\alpha=1}^{2n}(\lambda - \mu_\alpha)}. \tag{10}$$

In this article we concentrate on the case with $m = n$ (without double points, i.e. $M(\lambda)$ is a constant). Thus the full set of moduli is: $t_I \equiv \{S_i, \ t_0, \ t_k\}$, $i = \overline{1, n-1}$, $k = \overline{1, n}$, where occupancy numbers S_i are defined as integrals over A-cycles on the curve y,

$$S_i \equiv \frac{1}{4\pi i} \oint_{A_i} y d\lambda \tag{11}$$

To construct $\mathcal{F}_1$, we also define the polynomials $H_I(\lambda)$

$$\frac{dy}{dt_I} = \frac{H_I(\lambda)}{y(\lambda)} \tag{12}$$

and matrix $\sigma_{i,j}$

$$\sigma_{j,i} \equiv \oint_{A_j} \frac{\lambda^{i-1}}{y(\lambda)} d\lambda, \quad i, j = \overline{1, n-1}. \tag{13}$$

It can be shown [6] that for polynomials $H_k(\lambda) \equiv \sum_{l=1}^{n-1} H_{l,k} \lambda^{l-1}$, $k = \overline{1, n-1}$ corresponded S_k,

$$\sum_{l=1}^{n-1} \sigma_{j,l} H_{l,k} = \delta_{j,k} \quad \text{for} \quad j, k = \overline{1, n-1}. \tag{14}$$

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2 Two-cut case

According to paper [5] the holomorphic part of the genus one B-model amplitude is, up to an additive constant,

$$\mathcal{F}_1 = \frac{1}{2} \log \left(\det \left(\frac{\partial \mu_i^-}{\partial S_j} \right) \Delta^{2/3} \frac{2}{\mu_2^+ - \mu_1^+} \right), \tag{15}$$

where $\mu_1^- = \frac{1}{2}(\mu_1 - \mu_2)$, $\mu_2^- = \frac{1}{2}(\mu_3 - \mu_4)$, $\mu_1^+ = \frac{1}{2}(\mu_1 + \mu_2)$ and $\mu_2^+ = \frac{1}{2}(\mu_3 + \mu_4)$. On the other hand there is an answer for $\mathcal{F}_1$ obtained directly from solving the loop equations (7) for matrix model [9, 10], or using conformal field theory technique [11, 12]

$$\mathcal{F}_1 = -\frac{1}{24} \log \left(\prod_{\alpha=1}^{2n} M(\mu_\alpha) \cdot \Delta^4 \cdot (\det_{i,j} \sigma_{j,i})^{12} \right), \tag{16}$$

which, in the two-cut case without double points reads as

$$\mathcal{F}_1 = -\frac{1}{24} \log \left(\Delta^4 \cdot \sigma^{12} \right), \tag{17}$$

σ (13) here is 1×1 matrix. To obtain (15) from (17), one should prove the following formula

$$\det \left(\frac{\partial \mu_i^-}{\partial S_j} \right) \Delta \frac{2}{\mu_3 + \mu_4 - \mu_1 - \mu_2} \sigma = 1. \tag{18}$$

We can explicitly find the derivatives $\frac{\partial S_i}{\partial \mu_j^-}$ (instead of $\frac{\partial \mu_i^-}{\partial S_j}$), keeping times t_k constant. To do so one should first write $\frac{\partial S_i}{\partial \mu_j}$ then make the change of variables from $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ to $\{t_1, t_2, \mu_1^-, \mu_2^-\}$. Then

$$\frac{\partial S_i}{\partial \mu_j^-} = \frac{\partial S_i}{\partial \mu_k} \frac{\partial \mu_k}{\partial \mu_j^-}, \quad i, j = \overline{1, n}, \ k = \overline{1, 2n}. \tag{19}$$

$\frac{\partial \mu_k}{\partial \mu_j^-}$ here are obtained by inverting the matrix $\left(\frac{\partial \mu_j^-}{\partial \mu_k}, \frac{\partial S_j}{\partial \mu_k} \right)$. After this, it is easy to rewrite (18) using the elliptic integrals:

$$\sqrt{\frac{\mu_4 - \mu_2}{\mu_3 - \mu_1}} \left(\frac{\mu_4 - \mu_1}{\mu_4 - \mu_2} \Pi \left(-\frac{\mu_2 - \mu_1}{\mu_4 - \mu_2}, \kappa \right) + \right.$$
$$\left. \frac{\mu_3 - \mu_2}{\mu_4 - \mu_2} \Pi \left(\frac{\mu_4 - \mu_3}{\mu_4 - \mu_2}, \kappa \right) + K(\kappa) \right) = \frac{\pi}{2}. \tag{20}$$

where $\kappa = \sqrt{\frac{(\mu_2 - \mu_1)(\mu_4 - \mu_3)}{(\mu_4 - \mu_2)(\mu_3 - \mu_1)}}$, $\Pi(\nu, \kappa)$ and $K(\kappa)$ are complete elliptic integrals of the third and first kinds respectively. To prove this statement, one can rewrite the elliptic integrals of the third kind via the complete and incomplete elliptic integrals of the first and the second kinds (these formulas can be found in [13] (formulas 22, 24 from chapter 13.8); note, however, that in [13] there is a misprint in these formulas)

$$k'^2 \frac{\sin \theta \cos \theta}{\sqrt{1 - k'^2 \sin \theta^2}} [\Pi(1 - k'^2 \sin \theta^2, \kappa) - K(\kappa)] =$$
$$\frac{\pi}{2} - (E(\kappa) - K(\kappa))F(\sin \theta, k') - K(\kappa)E(\sin \theta, k') \tag{21}$$

$$\frac{\sqrt{1 - k'^2 \sin \theta^2}}{\sin \theta \cos \theta} [\Pi(-k'^2 \tan \theta^2, \kappa) - K(\kappa) \cos \theta^2] =$$
$$(E(\kappa) - K(\kappa))F(\sin \theta, k') - K(\kappa)E(\sin \theta, k') \tag{22}$$

where $k' = \sqrt{1 - k^2}$, $\theta \in [0, \pi/2]$. In this case one should put $\sin^2 \theta = \frac{\mu_3 - \mu_1}{\mu_4 - \mu_1}$.

The same computation can be done for any other partition of μ_i into the two sets $\mu_{1,2}^\pm$ (without changing σ), say, for $\mu_1^- = \frac{1}{2}(\mu_1 - \mu_3)$, $\mu_2^- = \frac{1}{2}(\mu_2 - \mu_4)$, $\mu_1^+ = \frac{1}{2}(\mu_1 + \mu_3)$ and $\mu_2^+ = \frac{1}{2}(\mu_2 + \mu_4)$. It leads to the same result (15), however, the perturbative calculation in this case is irrelevant.

3 Generalization for n-cut solution

A natural generalisation of (15) is

$$\mathcal{F}_1 = \frac{1}{2}\log\left(\det\left\|\frac{\partial\{\mu_j^-\}}{\partial\{S_i, S_n\}}\right\| \Delta^{2/3}\Delta^{-1}(\mu_j^+)\right),\tag{23}$$

where we divided all the branching points into two ordered sets $\{\mu_j^{(1)}\}_{j=1}^n$ and $\{\mu_j^{(2)}\}_{j=1}^n$ and performed a linear orthogonal transformation of $\mu_j^{(1,2)}$ to the quantities $\{\mu_j^+\}$ and $\{\mu_j^-\}$, $j = \overline{1, n}$,

$$\mu_j^\pm = \mu_j^{(1)} \pm \mu_j^{(2)}.\tag{24}$$

To prove formula (23), one should calculate the derivative of the branching points μ_j with respect to the moduli $\{t_K\} \equiv \{S_1..S_{n-1}, t_0..t_n\}$ [6]:

$$\frac{\partial\mu_\alpha}{\partial t_K} = \frac{H_K(\mu_\alpha)}{M(\mu_\alpha)\prod_{\beta\neq\alpha}(\mu_\alpha - \mu_\beta)}.\tag{25}$$

The polynomials $H_I(\lambda)$ corresponding to the variables t_k, $k \geq 1$ always have the coefficient k at the highest term λ^{n-1+k} and the polynomial corresponding to t_0 starts with λ^{n-1}. Therefore, one can find the determinant:

$$\det\left\|\frac{\partial\{\mu_{\alpha_j}\}}{\partial\{S_i, S_n, t_k\}}\right\| = \frac{\Delta(\mu_{\alpha_j})\cdot\left(\det_{l,m}\sigma_{l,m}\right)^{-1}}{\prod_{i=1}^{2n} M(\mu_{\alpha_i})\prod_{j=1}^{2n}\left(\prod_{\beta\neq\alpha_j}^{2n}(\mu_{\alpha_j} - \mu_\beta)\right)}\tag{26}$$

Indeed, consider the left hand side of (26).

$$\det\left\|\frac{\partial\{\mu_{\alpha_j}\}}{\partial\{S_i, S_n, t_k\}}\right\| = \frac{\left(\det_{K,j} H_K(\mu_{\alpha_j})\right)}{\prod_{i=1}^{2n} M(\mu_{\alpha_i})\prod_{j=1}^{2n}\left(\prod_{\beta\neq\alpha_j}^{2n}(\mu_{\alpha_j} - \mu_\beta)\right)}\tag{27}$$

The change of variables $\{S_1, .., S_n\} \rightarrow \{S_1, ..., S_{n-1}, t_0\}$ does not change the determinant. To obtain the Vandermonde determinant in the right hand side of (26), there should be, instead of the polynomials H_K, polynomials of degree $2n - i + 1$ where i is the line number, with unit leading coefficients. To this end, one should multiply the matrix $H_K(\mu_{\alpha_j})$ with the block diagonal matrix

$$\tilde{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}.\tag{28}$$

This gives the factor $(\det\tilde{\sigma})^{-1} = (\det_{l,m}\sigma_{l,m})^{-1}$. Lines from 1 to $n + 1$ contribute to $n!$ which could be omitted from the free energy. The Vandermonde determinant $\Delta(\mu_{\alpha_j})$ then combines with the rational factors in the denominator to produce $(-1)^{\sum_{j=1}^n \alpha_j}\Delta(\overline{\mu_{\alpha_j}})/\Delta(\mu)$, where $\Delta(\overline{\mu_{\alpha_j}})$ is the Vandermonde determinant for the supplementary set of n branching points not entering the set $\{\mu_{\alpha_j}\}_{j=1}^n$ whereas $\Delta(\mu)$ is the total Vandermonde determinant. Now we should put $M(\mu_\alpha)$ constant independent of α. Expanding the determinant in (23) by each line and neglecting the additive constant $\frac{1}{2}\log 2^n$, one obtain (16).

Introducing the quantities

$$\phi_I^\alpha \equiv \frac{H_I(\mu_\alpha)}{M^{1/3}(\mu_\alpha)\prod_{\beta\neq\alpha}(\mu_\alpha - \mu_\beta)^{2/3}},\tag{29}$$

one can rewrite (16) in a more simple form:

$$\mathcal{F}_1 = \frac{1}{2}\log\left(\det_{I,\alpha}\phi_I^\alpha\right)\tag{30}$$

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4   Perturbative Formula

We have also performed the perturbative check of (23) for the 3-cut case. It is easier to make the expansion not in the moduli S_i but in the difference of the branching points μ_j^-. In order to calculate $\det \left\| \frac{\partial \{S_i, S_n\}}{\partial \{\mu_j^-\}} \right\|$, one should rewrite S_i and $\sigma_{i,j}$ in terms of μ_i^+, μ_j^- and expand them in μ_i^-

$$S_l = \frac{1}{2} res_{\lambda=\mu_l^+} \prod_{i=1}^{n} \sum_{k=0}^{\infty} \frac{(\mu_i^-)^{2k} c_k}{(\lambda - \mu_i^+)^{2k-1}} \tag{31}$$

$$\sigma_{l,j} = \frac{1}{2} res_{\lambda=\mu_l^+} \lambda^{j-1} \prod_{i=1}^{n} \sum_{k=0}^{\infty} \frac{(\mu_i^-)^{2k} \tilde{c}_k}{(\lambda - \mu_i^+)^{2k+1}} \tag{32}$$

c_k and $\tilde{c}_k$ are the Taylor coefficients for $\sqrt{1-x}$ and $\frac{1}{\sqrt{1-x}}$ respectively. It should be mentioned that derivatives $\frac{\partial \{S_i\}}{\partial \{\mu_j^-\}}$ are taken at t_k constant, while in (31) S_k are functions of μ^+, μ^-. This problem is solved by calculating the transition matrix from $\{\mu_k^-, t_k\}$ to $\{\mu_k^-, \mu_k^+\}$ and inverting it. We have done this calculation up to $(\mu^-)^3$ and found it in perfect agreement with (16) (up to an additive constant mentioned).

Acknowledgments

References

[1] F. Cachazo, Kenneth A. Intriligator, and Cumrun Vafa. A large n duality via a geometric transition. *Nucl. Phys.*, B603:3–41, 2001.

[2] Robbert Dijkgraaf and Cumrun Vafa. Matrix models, topological strings, and supersymmetric gauge theories. *Nucl. Phys.*, B644:3–20, 2002.

[3] Robbert Dijkgraaf and Cumrun Vafa. On geometry and matrix models. *Nucl. Phys.*, B644:21–39, 2002.

[4] Robbert Dijkgraaf and Cumrun Vafa. A perturbative window into non-perturbative physics. *hep-th/0208048*, 2002.

[5] Albrecht Klemm, Marcos Marino, and Stefan Theisen. Gravitational corrections in supersymmetric gauge theory and matrix models. *JHEP*, 03:051, 2003.

[6] L. Chekhov, A. Marshakov, A. Mironov, and D. Vasiliev. Complex geometry of matrix models. *hep-th/0506075*, 2005.

[7] Yu. Makeenko. Loop equations in matrix models and in 2-d quantum gravity. *Mod. Phys. Lett.*, A6:1901–1913, 1991.

[8] A. A. Migdal. Loop equations and 1/n expansion. *Phys. Rept.*, 102:199–290, 1983.

[9] G. Akemann. Higher genus correlators for the hermitian matrix model with multiple cuts. *Nucl. Phys.*, B482:403–430, 1996.

[10] L. Chekhov. Genus one correlation to multi-cut matrix model solutions. *Theor. Math. Phys.*, 141:1640–1653, 2004.

[11] Ivan K. Kostov. Conformal field theory techniques in random matrix models. *hep-th/9907060*, 1999.

[12] Robbert Dijkgraaf, Annamaria Sinkovics, and Mine Temurhan. Matrix models and gravitational corrections. *Adv. Theor. Math. Phys.*, 7:1155–1176, 2004.

[13] Erdélyi A. Bateman H. *Higher transcendental functions. Vol. III.* New York: McGraw-Hill Book Co., 1955. Bateman Manuscript Project, California Institute of Technology.